A case study in double categories

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Part I

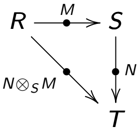
Morphisms

Ring

- The category **Ring** has (not necessarily commutative) rings with 1 as objects and homomorphisms preserving 1 as morphisms
- This is a very good category. It's monadic over **Set**, so complete and cocomplete. It's locally finitely presentable, etc.
- So why mess with it?

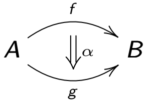
Bimodules

- Given rings R and S, an S-R-bimodule M is simultaneously a left S-module and a right R-module whose left and right actions commute
$$(sm)r = s(mr)$$
- If T is another ring and N a T-S-bimodule, the tensor product over S, $N \otimes_S M$ is naturally a T-R-bimodule. We have associativity isomorphisms
$$P \otimes_T (N \otimes_S M) \cong (P \otimes_T N) \otimes_S M$$
and unit isomorphisms
$$M \otimes_R R \cong M \cong S \otimes_S M$$
- To keep track of the various rings involved and what's acting on what and on which side we can write
$$M : R \xrightarrow{\bullet} S$$
to mean that M is an S-R-bimodule
- The tensor product looks like a composition



Bicategories

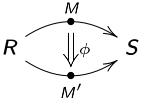
- Rings with bimodules as morphisms is not a category but a *bicategory*, $\mathcal{B}im$
- In a bicategory we have objects and morphisms which compose, but composition is only associative and unitary up to isomorphism
- To express this isomorphism we need morphisms between morphisms



called 2-*cells*

$\mathcal{B}im$

- In our example $\mathcal{B}im$
  - Objects are rings
  - Morphisms (1-cells) are bimodules
  - A 2-cell



  is a linear map of bimodules, i.e. a function such that

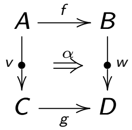
$$\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$$
$$\phi(sm) = s\phi(m)$$
$$\phi(mr) = \phi(m)r$$

- $\mathcal{B}im$ is a very good bicategory
  - Cartesian bicategory
  - Biclosed

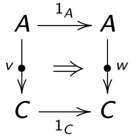
$$\begin{array}{c} M \longrightarrow N \obslash_T P \\ \hline N \otimes_S M \longrightarrow P \\ \hline N \longrightarrow P \oslash_R M \end{array}$$

Double categories

- A *double category* $\mathbb{A}$ has objects (A, B, C, D below) and *two* kinds of morphism, strong, which we call *horizontal* (f, g below) and *weak*, or *vertical* (v, w below) These are related by a further kind of morphism, double cells as in



- The horizontal arrows form a category **Hor**$\mathbb{A}$ with composition denoted by juxtaposition and identities by 1_A. Cells can also be composed horizontally forming a category
- The vertical arrows compose to give a bicategory $\mathcal{V}ert\mathbb{A}$ whose 2-cells are the *globular cells* of $\mathbb{A}$, i.e. those with identities on the top and bottom



Vertical composition is denoted by $\bullet$ and vertical identities by id_A

$\mathbb{R}\mathrm{el}$

Example

$\mathbb{R}\mathrm{el}$ has sets as objects and functions as horizontal arrows, so $\mathbf{Hor}\mathbb{R}\mathrm{el} = \mathbf{Set}$. A vertical arrow $R : X \longrightarrow\!\!\!\!\!\!\bullet\;\; Y$ is a relation between X and Y and there is a unique cell

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle R}\,\big\downarrow\!\bullet & \Rightarrow & \bullet\!\big\downarrow\,{\scriptstyle R'} \\ Y & \xrightarrow[g]{} & Y' \end{array}$$

if (and only if) we have

$$\forall_{x,y}(x \sim_R y \Rightarrow f(x) \sim_{R'} g(y))$$

The double category $\mathbb{R}\text{ing}$

- Objects are rings
- Horizontal arrows are homomorphisms
- Vertical arrows are bimodules
- A double cell

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle M}\big\downarrow\!\bullet & \overset{\phi}{\Longrightarrow} & \bullet\!\big\downarrow{\scriptstyle M'} \\ S & \xrightarrow[g]{} & S' \end{array}$$

  is a linear map in the sense that it preserves addition and is compatible with the actions

$$\phi(sm) = g(s)\phi(m)$$

$$\phi(mr) = \phi(m)f(r)$$

- Vertical composition is $\otimes$

Companions

- Let $\mathbb{A}$ be a double category, $f : A \longrightarrow B$ a horizontal arrow, and $v : A \longrightarrow\!\!\bullet\!\!\longrightarrow B$ a vertical one in $\mathbb{A}$. We say that v is a *companion* of f if we are given cells, the *binding cells*, α and β, such that

$$
\begin{array}{ccccc}
A & \xrightarrow{1_A} & A & \xrightarrow{f} & B \\
{\scriptstyle \mathrm{id}_A}\downarrow\bullet & \alpha & \bullet\downarrow{\scriptstyle v} & \beta & \bullet\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[f]{} & B & \xrightarrow[1_B]{} & B
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \mathrm{id}_A}\downarrow\bullet & \mathrm{id}_f & \bullet\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[f]{} & B'
\end{array}
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{\scriptstyle v}\downarrow\bullet & 1_v & \bullet\downarrow{\scriptstyle v} \\
B & \xrightarrow[1_B]{} & B
\end{array}
$$

  Companions, when they exist, are unique up to isomorphism, and we use the notation f_* to denote a choice of companion for f
- In $\mathbb{R}\mathrm{el}$, every function $f : A \longrightarrow B$ has a companion, viz. its graph $Gr(f) \subseteq A \times B$

Companions in $\mathbb{R}\text{ing}$

Proposition

(a) In $\mathbb{R}\text{ing}$, every homomorphism $f : R \longrightarrow S$ has a companion, namely S considered as an S-R-bimodule with actions $\diamond$ given by

$$\begin{aligned} s' \diamond s &= s's \\ s \diamond r &= sf(r) \end{aligned}$$

(b) A bimodule $M : R \longrightarrow\!\!\!\!\bullet\!\!\!\!\longrightarrow S$ is a companion, i.e. is of the form f_* for some horizontal arrow f, if and only if it is free of rank 1 as a left S-module

(c) Homomorphisms corresponding to different free generators are related by conjugation by a unit of S

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So $f(r_1 + r_2) = f(r_1) + f(r_2)$

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Conjoints

- Let $f : A \longrightarrow B$ be a horizontal arrow in a double category $\mathbb{A}$ and $v : B \longrightarrow\!\!\!\!\!\!\bullet\;\; A$ a vertical one. We say that v is *conjoint* to f if we are given cells ψ and χ (*conjunctions*) such that

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{1_B} & B \\
{\scriptstyle \mathrm{id}_A}\big\downarrow & \overset{\psi}{\Longrightarrow} & \big\downarrow{\scriptstyle v} & \overset{\chi}{\Longrightarrow} & \big\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[1_A]{} & A & \xrightarrow[f]{} & B
\end{array}
\quad = \quad
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A & \xrightarrow[f]{} & B
\end{array},
$$

$$
\begin{array}{ccc}
B & \xrightarrow{1_B} & B \\
{\scriptstyle v}\big\downarrow & \overset{\chi}{\Longrightarrow} & \big\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[f]{} & B \\
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\end{array},
$$

- In $\mathbb{R}\mathrm{ing}$, every homomorphism $f : R \longrightarrow S$ has a conjoint f^*, namely $S : S \longrightarrow\!\!\!\!\!\!\bullet\;\; R$ with left action by R given by “restriction”

$$r \diamond s = f(r)s$$

Rank 2

- Homomorphisms $f\colon R \longrightarrow S$ correspond to bimodules $M\colon R \longrightarrow\!\!\bullet\!\!\longrightarrow S$ which are free on one generator as left S-modules
- What if M is free on 2 generators?
- Assume M free on m_1, m_2 as a left S-module. Nothing is said about the right action (as before). Then for each $r \in R$ we get unique $s_{11}, s_{12}, s_{21}, s_{22} \in S$ such that

$$\begin{array}{rcccc} m_1 r & = & s_{11} m_1 & + & s_{12} m_2 \\ m_2 r & = & s_{21} m_1 & + & s_{22} m_2 \end{array}$$

  Let's denote s_{ij} by $f_{ij}(r)$. So to each r we associate not 2 but 4 elements of S or rather a 2×2 matrix in S

Rank p

If M is free on p generators $m_1, \ldots, m_p$:

$$m_i r = \sum_{j=1}^{p} f_{ij}(r) m_j$$

Theorem

(a) Any matrix-valued homomorphism $f : R \longrightarrow Mat_p(S)$ *induces an* S-R*-bimodule structure on* $S^{(p)}$
(b) Any S-R*-bimodule* $M : R \longrightarrow\!\!\!\bullet\!\!\!\!\longrightarrow S$ *which is free on* p *generators as a left* S*-module is isomorphic (as on* S-R*-bimodule) to* $S^{(p)}$ *with* R*-action induced by a homomorphism* $f : R \longrightarrow Mat_p(S)$ *as in (a)*
(c) Homomorphisms corresponding to different free generators are related by conjugation by an invertible $p \times p$ *matrix* A *in* $Mat_p(S)$

Example

(Pairs of homomorphisms)
Let $f, g : R \longrightarrow S$ be homomorphisms. Then we get a homomorphism $h : R \longrightarrow Mat_2(S)$ given by

$$h(r) = \begin{bmatrix} f(r) & 0 \\ 0 & g(r) \end{bmatrix}$$

Example

(Derivations)
Let $f : R \longrightarrow S$ be a homomorphism and d an f-derivation, i.e. an additive function $d : R \longrightarrow S$ such that

$$d(rr') = d(r)f(r') + f(r)d(r')$$

Then we get a homomorphism $R \longrightarrow Mat_2(S)$

$$r \longmapsto \begin{bmatrix} f(r) & 0 \\ d(r) & f(r) \end{bmatrix}$$

Example

More generally we can consider the subring of lower triangular matrices

$$L = \left\{ \begin{bmatrix} s & 0 \\ s' & s'' \end{bmatrix} \middle| s, s', s'' \in S \right\}$$

Then a homomorphism $R \longrightarrow Mat_2(S)$ that factors through L corresponds to a pair of homomorphisms $f, g : R \longrightarrow S$ and a derivation d from f to g, i.e. an additive function $d : R \longrightarrow S$ such that

$$d(rr') = d(r)f(r') + g(r)d(r')$$

A graded category of rings

- Homomorphisms $f\colon R \longrightarrow Mat_p(S)$ and $g\colon S \longrightarrow Mat_q(T)$ correspond to bimodules

$$S^{(p)}\colon R \longrightarrow\!\!\!\!\!\bullet\;\; S \quad \text{and} \quad T^{(q)}\colon S \longrightarrow\!\!\!\!\!\bullet\;\; T,$$

  and we can compose these

$$T^{(q)} \otimes_S S^{(p)} \cong T^{(pq)}$$

- This gives a composite gf

$$R \xrightarrow{f} Mat_p(S) \xrightarrow{Mat_p(g)} Mat_p Mat_q(T) \cong Mat_{pq}(T)$$

Thus we first apply f to an element $r \in R$ to get a $p \times p$ matrix in S, and then apply g to each entry separately to get a $p \times p$ block matrix of $q \times q$ matrices, and then consider this as a $(pq) \times (pq)$ matrix

Theorem

With this composition we get an $(\mathbb{N}^+, \cdot)$-graded category **Matring** *whose objects are rings and whose morphisms of degree p are homomorphisms into $p \times p$ matrices:*

$$\frac{R \xrightarrow{(p,f)} S \quad \text{in} \quad \textbf{Matring}}{f\colon R \longrightarrow Mat_p(S) \quad \text{in} \quad \textbf{Ring}}$$

The graded double category of rings

The double category $\mathbb{M}\mathrm{atring}$

- Objects rings
- Horizontal arrows $(p, f)\colon R \longrightarrow R'$
- Vertical arrows are bimodules $M\colon R \longrightarrow\!\!\!\!\bullet\!\!\!\!\longrightarrow S$
- A double cell

$$\begin{array}{ccc} R & \xrightarrow{(p,f)} & R' \\ {\scriptstyle M}\downarrow\bullet & \overset{\phi}{\Longrightarrow} & \bullet\downarrow{\scriptstyle M'} \\ S & \xrightarrow[(q,g)]{} & S' \end{array}$$

is a linear map (a cell in $\mathbb{R}\mathrm{ing}$)

$$\begin{array}{ccc} R & \xrightarrow{f} & Mat_p(R') \\ {\scriptstyle M}\downarrow\bullet & \overset{\phi}{\Longrightarrow} & \bullet\downarrow{\scriptstyle Mat_{q,p}(M')} \\ S & \xrightarrow[g]{} & Mat_q(S') \end{array}$$

where $Mat_{q,p}(M')$ is the bimodule of $q \times p$ matrices with entries in M', with the $Mat_q(S')$ action given by matrix multiplication on the left, and similarly for $Mat_p(R')$

Properties of $\mathbb{M}\text{atring}$

Theorem

(1) $\mathbb{M}\text{atring}$ *is a double category*

(2) Every horizontal arrow has a companion

(3) Every horizontal arrow has a conjoint

(4) The vertically full double subcategory determined by the morphisms of degree 1 *is isomorphic to* $\mathbb{R}\text{ing}$

Cauchy completeness

- If a horizontal arrow $f\colon A \longrightarrow B$ in a double category $\mathbb{A}$ has a companion f_* and a conjoint f^* then f_* is left adjoint to f^* in $\mathcal{V}ert\mathbb{A}$
- Say that B is *Cauchy complete* if every adjoint pair $v \dashv u$, $v\colon A \longrightarrow\!\!\bullet\!\!\longrightarrow B$, $u\colon B \longrightarrow\!\!\bullet\!\!\longrightarrow A$ is of the form $f_* \dashv f^*$ for some $f\colon A \longrightarrow B$
- $\mathbb{A}$ is *Cauchy* if every object is Cauchy complete

Example

$\mathbb{R}\mathrm{el}$ is Cauchy

Characterization for bimodules

The following theorem is well-known

Theorem

A bimodule $M : R \longrightarrow\!\!\!\!\!\!\bullet\;\; S$ has a right adjoint in $\mathcal{B}im$ if and only if it is finitely generated and projective as a left S-module

Finitely generated projective

M is finitely generated, by $m_1, \dots, m_p$ say, if and only if the S-linear map

$$\tau : S^{(p)} \longrightarrow M$$

$\tau(s_1 \dots s_p) = \sum_{i=1}^{p} s_i m_i$ is surjective. If M is S-projective, then τ splits, i.e. there is an S-linear map

$$\sigma : M \longrightarrow S^{(p)}$$

such that $\tau\sigma = 1_M$. In fact, M is a finitely generated and projective S-module if and only if there exist p, τ, σ such that $\tau\sigma = 1_M$

Let the components of σ be $\sigma_1, \dots, \sigma_p : M \longrightarrow S$. Then $\tau\sigma = 1_M$ means that for every $m \in M$ we will have

$$m = \sum_{i=1}^{p} \sigma_i(m) m_i$$

i.e. the σ_i provide an S-linear choice of coordinates for m relative to the generators $m_1 \dots m_p$. All of this is independent of R

Non-unital homomorphisms

For any r we can write

$$m_i r = \sum_{j=1}^{p} \sigma_j(m_i r) m_j$$

If we let $f_{ij}(r) = \sigma_j(m_i r)$ we get the same formula as for $\mathbb{M}\mathrm{atring}$ (on frame 15)

$$m_i r = \sum_{j=1}^{p} f_{ij}(r) m_j$$

Theorem

(1) The functions f_{ij} define a non-unital homomorphism $f : R \longrightarrow Mat_p(S)$
(2) Any such homomorphism comes from a bimodule which is finitely generated and projective as a left S-module
(3) Two representations (p, f) and (q, g) of the same S-R-bimodule (finitely generated projective over S) are related as follows: there is a $q \times p$ matrix A and a $p \times q$ matrix B, both with entries in S, such that

(a) $Af(1) = A$ *and* $Af(r) = g(r)A$

(b) $Bg(1) = B$ *and* $Bg(r) = f(r)B$

(c) $AB = g(1)$ *and* $BA = f(1)$

Amplifying homomorphisms

Non-unital homomorphisms $R \longrightarrow Mat_p(S)$ have already appeared in the quantum field theory literature

(see e.g. Szlachanyi, K, Vecsernyes, K, Quantum symmetry and braid group statistics in G-spin models, Commun. Math. Phys. 156, 127-168 (1993)) where they are called *amplifying homomorphisms* or *amplimorphisms* for short

The double category $\mathbb{A}\text{mpli}$

- Objects: rings
- Horizontal arrows: amplimorphisms $R \longrightarrow S$,
- Vertical arrows: bimodules $M : R \longrightarrow\!\!\!\!\!\bullet\;\; S$
- Cells:

$$\begin{array}{ccc} R & \xrightarrow{(p,f)} & R' \\ {\scriptstyle M}\downarrow\bullet & \overset{\phi}{\Longrightarrow} & \bullet\downarrow{\scriptstyle M'} \\ S & \xrightarrow[(q,g)]{} & S' \end{array} \qquad \text{are cells} \qquad \begin{array}{ccc} R & \xrightarrow{f} & Mat_p(R') \\ {\scriptstyle M}\downarrow\bullet & \overset{\phi}{\Longrightarrow} & \bullet\downarrow{\scriptstyle Mat_{q,p}(M')} \\ S & \xrightarrow[g]{} & Mat_q(S') \end{array}$$

i.e. additive functions $\phi : M \longrightarrow Mat_{q,p}(M')$ such that

$$\phi(mr) = \phi(m)f(r) \qquad \phi(sm) = g(s)\phi(m)$$

Theorem

(1) $\mathbb{A}\text{mpli}$ is a double category
(2) $\mathbb{A}\text{mpli}$ is vertically self dual
(3) Every horizontal arrow has a companion and a conjoint
(4) $\mathbb{A}\text{mpli}$ is Cauchy

Part II

Functors

Monadic

Ring is monadic over **Set**

$U\colon \mathbf{Ring} \longrightarrow \mathbf{Set}$ Forgetful

$F\colon \mathbf{Set} \longrightarrow \mathbf{Ring}$ Free

$F(X) = \mathbb{Z}\{X\}$ = Ring of polynomials with integer coefficients in the non-commuting variables $x, y, z ... \in X$

$F \dashv U$

Gives a monad $T = UF$ on **Set**

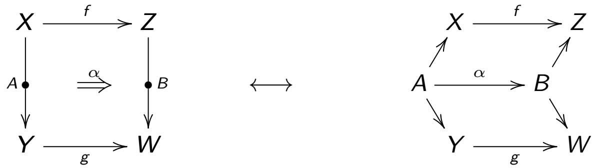
Ring is the category of algebras for T

Can we extend this to $\mathbb{R}\mathrm{ing}$?

$\mathbb{S}\mathrm{et}$

$\mathbb{S}\mathrm{et}$ is the double category of sets

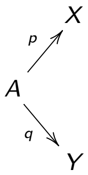
- Objects are sets
- Horizontal arrows are functions
- Vertical arrows are *spans*
- Cells are span morphisms



- Vertical composition uses pullbacks

Spans

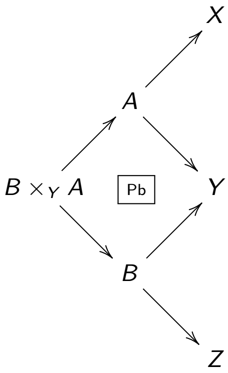
A span is to be thought of as a constructive or intensional relation
Suppose we have a span



How can $x \in X$ be related to $y \in Y$?
If there's an $a \in A$ such that $x = p(a)$ and $y = q(a)$
a is the reason (or proof) that x is related to y

$$x \sim_a y$$

Vertical composition



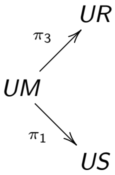
How can $x \in X$ be related to $z \in Z$?

There should be a y and "reasons" a and b such that

$$x \sim_a y \quad \text{and} \quad y \sim_b z$$

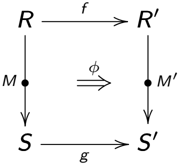
Hence the pullback

$UR =$ Underlying set of R
$Uf =$ Underlying function of f
For a bimodule $M\colon R \longrightarrow\!\!\!\bullet\!\!\!\longrightarrow S$



$$UM = \{(s, m, r) | sm = mr\}$$

For a cell



$$U\phi\colon UM \longrightarrow UM'$$
$$(s, m, r) \mapsto (gs, \phi m, fr)$$

Why this U?

Recall: If M is free over S on one generator m, it induces a homomorphisms $f : R \longrightarrow S$. $f(r)$ is the unique s such that

$$sm = mr$$

If m is not a free generator we just get a relation, but a constructive one: m is the reason that s is related to r

$$s \sim_m r \iff sm = mr$$

Note that $\sim_m$ is a "ring congruence"

- $s \sim_m r \;\;\&\;\; s' \sim_m r' \Rightarrow s + s' \sim_m r + r'$
- $s \sim_m r \;\;\&\;\; s' \sim_m r' \Rightarrow ss' \sim_m rr'$
- $s \sim_m r \;\;\&\;\; s \sim_m r' \;\;\&\;\; s' \sim_m r' \Rightarrow s' \sim_m r$

Preservation of composition

U preserves horizontal composition of arrows and cells
But it doesn't preserve vertical composition!

Consider $R \xrightarrow[\bullet]{M} S \xrightarrow[\bullet]{N} T$

$$U(N \otimes_S M) = \{(t, \sum n_i \otimes m_i, r) | \sum tn_i \otimes m_i = \sum n \otimes m_i r\}$$

$$U(N) \times_{U(S)} U(M) = \{(t, n, s, m, r) | tn = ns \quad \& \quad sm = mr\}$$

We have a comparison morphism

$$\Upsilon_2 \colon U(N) \times_{U(S)} U(M) \longrightarrow U(N \otimes_S M)$$

$$(t, n, s, m, r) \longmapsto (t, n \otimes m, r)$$

There's also a comparison

$$\Upsilon_0 \colon \mathrm{Id}_{UR} \longrightarrow U(\mathrm{Id}_R)$$

$$r \longmapsto (r, 1, r)$$

Lax double functors

U is a *lax double functor*

- Υ_0 and Υ_2 are horizontally natural
- Satisfy associativity and unit conditions, formally the same as for lax functors of bicategories

Adjoints to lax double functors

Given a lax double functor $U \colon \mathbb{B} \longrightarrow \mathbb{A}$ what does it mean for it to have a left adjoint?

- U has to have a left adjoint F at the level of objects and horizontal arrows

$$\frac{FA \longrightarrow B}{A \longrightarrow UB}$$

- U has to have a left adjoint at the level of vertical arrows and cells

$$\begin{array}{ccc} FA & \longrightarrow & B \\ {\scriptstyle Fv}\downarrow\!\bullet & \Rightarrow & \bullet\!\downarrow{\scriptstyle w} \\ FA' & \longrightarrow & B' \end{array} \qquad \Bigg| \qquad \begin{array}{ccc} A & \longrightarrow & UB \\ {\scriptstyle v}\downarrow\!\bullet & \Rightarrow & \bullet\!\downarrow{\scriptstyle Uw} \\ A' & \longrightarrow & UB' \end{array}$$

and the vertical domain and codomain of Fv must be FA and FA'
Then the mates calculus automatically makes F into an oplax double functor

This is the general situation for adjunctions $F \dashv U$ between double categories
$U \colon \mathbb{B} \longrightarrow \mathbb{A}$, $F \colon \mathbb{A} \longrightarrow \mathbb{B}$, F is oplax and U is lax

Adjoint to U

$U\colon \mathbb{R}\mathrm{ing} \longrightarrow \mathbb{S}\mathrm{et}$ does have a left adjoint F

$FX = \mathbb{Z}\{X\} =$ Free ring on X

$$\begin{array}{c} X \\ {\scriptstyle p}\big\uparrow \\ A \\ {\scriptstyle q}\big\downarrow \\ Y \end{array}$$

$F(p,q) = \mathbb{Z}\{Y\}A\mathbb{Z}\{X\}/qAp$

- $\mathbb{Z}\{Y\}A\mathbb{Z}\{X\}$ is the free $\mathbb{Z}\{Y\}$-$\mathbb{Z}\{X\}$ bimodule generated by A (finite sums of things like $(2y_1y_2 - y_2y_1 + y_3^3)\ a\ (x_2^2 + 3x_1x_3)$)
- qAp is the subbimodule generated by $\{q(a)a - ap(a) | a \in A\}$

F is truly oplax – not pseudo nor even normal

Monad?

Given an oplax-lax adjunction, the composite $T = UF$ is neither lax nor oplax

Enough structure is there to define algebras $TA \xrightarrow{a} A$, horizontal morphisms $(A, a) \xrightarrow{f} (C, c)$

Vertical morphisms

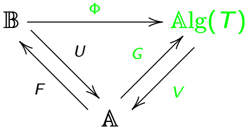
$$
\begin{array}{c}
(A, a) \\
{\scriptstyle (v,\alpha)} \downarrow \\
(B, b)
\end{array}
\qquad\qquad
\begin{array}{ccc}
TA & \xrightarrow{a} & A \\
{\scriptstyle Tv} \downarrow & \overset{\alpha}{\Rightarrow} & \downarrow {\scriptstyle v} \\
TB & \xrightarrow[b]{} & B
\end{array}
$$

and cells

$$
\begin{array}{ccc}
(A, a) & \xrightarrow{f} & (C, c) \\
{\scriptstyle (v,\alpha)} \downarrow & \overset{\gamma}{\Rightarrow} & \downarrow {\scriptstyle (w,\beta)} \\
(B, b) & \xrightarrow[g]{} & (D, d)
\end{array}
$$

$$
\begin{array}{ccccc}
TA & \xrightarrow{a} & A & \xrightarrow{f} & C \\
{\scriptstyle Tv} \downarrow & \overset{\alpha}{\Rightarrow} & \downarrow {\scriptstyle v} & \overset{\gamma}{\Rightarrow} & \downarrow {\scriptstyle w} \\
TB & \xrightarrow[b]{} & B & \xrightarrow[g]{} & D
\end{array}
=
\begin{array}{ccccc}
TA & \xrightarrow{Tf} & TC & \xrightarrow{c} & C \\
{\scriptstyle Tv} \downarrow & \overset{T\alpha}{\Rightarrow} & \downarrow {\scriptstyle Tw} & \overset{\beta}{\Rightarrow} & \downarrow {\scriptstyle w} \\
TB & \xrightarrow[Tg]{} & TD & \xrightarrow[d]{} & D
\end{array}
$$

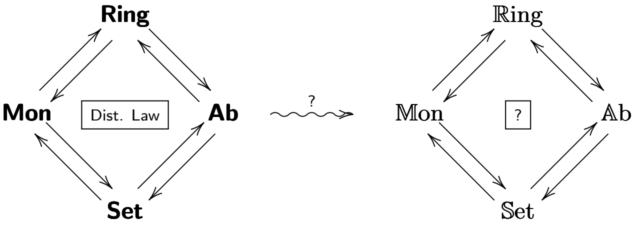
Horizontal morphisms can be composed and cells composed horizontally
But there is no way to compose vertical arrows! (Nor compose cells vertically)
There are forgetful V and free G and a comparison Φ:



They are horizontally functorial, but that's it
If we allow ourselves to use the F and U we can make $\mathbb{A}\mathrm{lg}(T)$ into a virtual double category

C.f. Cruttwell and Shulman, A unified framework for generalized multicategories, TAC Vol. 24 (2010)

1. Is there a workable theory for monads coming from oplax-lax adjunctions?
2. How do the double categories $\mathbb{M}\text{atring}$ and $\mathbb{A}\text{mpli}$ fit in?
3.



¡Gracias!